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On the preliminary study of a dielectric guide having a Piet Hein geometry

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Abstract : The wave equation and the field solutions have been derived for a new type of optical waveguide having a geometry like a Piet Hein curve. Such a waveguide may find applications in integrated optic technology.

Keywords : Electromagnetic wave propagation, optical waveguides

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The investigators have studied so far a wide variety of optical waveguides in respect of their cross sections that are indispensable in integrated optic technology. Among them the parabolic cylindrical waveguides and the guides with triangular cross sections are new [1–6]. Particularly the parabolic cylindrical waveguide geometry was reported first by Choudhury *et al* and several research papers have been appeared in the literature [3–6]. As rectangular and circular waveguides are indispensable in optical communication systems, a waveguide with a cross section which stands midway between the circle and rectangle would be rather interesting. An optical waveguide with its cross sectional geometry like a Piet Hein curve (Figure 1) just meets the situation, and would present some interesting

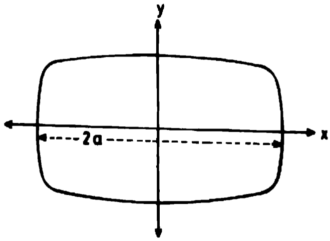

Figure 1. Cross section of the guide.

results. The specific property of such a structure will be that it will essentially possess the properties of both a circular waveguide and an elliptical waveguide. These guides would be highly advantageous because, due to specific structure of such guides, the undesirable effects (like wave scattering *etc*) are not expected to occur. These effects may exist in rectangular waveguides owing to the presence of sharp corners which affect partially the wave propagation. As non-circular waveguides are widely used in mode-detection, guides with a Piet Hein geometry, obviously, may also be used in optical detector circuits with more prominance. The basic requirement of a rigorous analysis of any type of waveguide structure is to investigate the nature of the wave equation and the wave function for the structure. The aim of the present communication is to derive the actual wave equation and the field functions for the optical waveguide having its geometry like a Piet Hein curve. The further rigorous analysis will be presented in a future communication.

The equation of Piet Hein curve in cartesian coordinate system is

$$x^4 + y^4 = a^4 \tag{1}$$

a being the principal semi-diameter of the curve. This equation can be transformed into the form

$$\frac{1}{x^2} - \frac{1}{y^2} = \frac{1}{c^2} \tag{2}$$

with c as a constant. From eqs. (1) and (2) it can be derived that

$$p+q = \left\{d+\left(d^2+b^2\right)^{1/2}\right\}^{1/2} - 4d^2 = A \text{ (say)} \tag{3}$$

$$p-q = d-\left(d^2+b^2\right)^{1/2} \tag{4}$$

and

$$pq = d\left(d^2+b^2\right)^{1/2} - d^2 \tag{5}$$

where $p = x^2, q = y^2, b = a^2$ and $d = c^2$. With the help of these equations one can evaluate

$$x = \left[\frac{1}{2}\left\{\left\{\left(c^2 + \sqrt{(a^4+c^4)}\right)^2 - 4c^4\right\}^{1/2} + c^2 - \sqrt{(a^4+c^4)}\right\}\right]^{1/2} \tag{6a}$$

$$y = \left[\frac{1}{2}\left\{\left\{\left(c^2 + \sqrt{(a^4+c^4)}\right)^2 - 4c^4\right\}^{1/2} - c^2 + \sqrt{(a^4+c^4)}\right\}\right]^{1/2} \tag{6b}$$

We now have to investigate whether the system (a, c, z) (Figure 2) is orthogonal; z-axis being the direction of propagation (perpendicular to the plane of paper). Using eqs. (6a) and (6b) we find

$$\frac{\partial x}{\partial a} = \frac{a^3}{x\sqrt{\left(c^4+a^4\right)}}\left(\frac{c^2-x^2}{A}\right) \tag{7a}$$

$$\frac{\partial y}{\partial a} = \frac{a^3}{y\sqrt{(c^4+a^4)}}\left(\frac{c^2+y^2}{A}\right) \tag{7b}$$

$$\frac{\partial x}{\partial c} = \frac{cy^2}{Ax\sqrt{(c^4+a^4)}}\left\{\sqrt{(a^4+c^4)}-c^2\right\} \tag{7c}$$

$$\frac{\partial y}{\partial c} = \frac{cx^2}{Ay\sqrt{(c^4+a^4)}}\left\{c^2-\sqrt{(a^4+c^4)}\right\} \tag{7d}$$

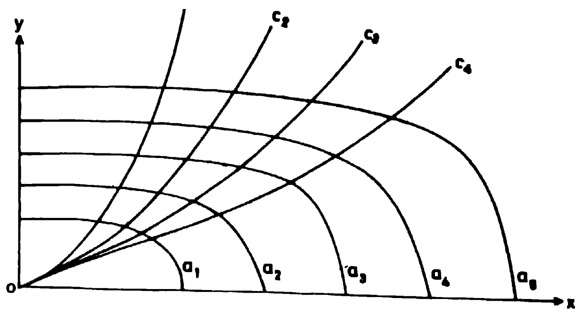

Figure 2. The new coordinate system (a, c, z)

Since $x = x\,(a, c)$ and $y = y\,(a, c)$, and therefore,

$$dx = \frac{\partial x}{\partial a}da + \frac{\partial x}{\partial c}dc$$

and
$$dy = \frac{\partial y}{\partial a}da + \frac{\partial y}{\partial c}dc$$

From eqs (7a)–(7d) it can be found that

$$\frac{\partial x}{\partial a}\,\frac{\partial x}{\partial c} + \frac{\partial y}{\partial a}\,\frac{\partial y}{\partial c} = 0 \tag{8}$$

Hence the system (a, c, z) is orthogonal.

Again using eqs. (6a) and (6b) it can be derived finally that

$$dx = \frac{a^3(c^2-x^2)}{Ax\sqrt{c^4+a^4}}\,da + \frac{cy^2}{Ax\sqrt{c^4+a^4}}\left\{\sqrt{(a^4+c^4)}-c^2\right\}dc \tag{9a}$$

$$dy = \frac{a^3(c^2+y^2)}{Ay\sqrt{c^4+a^4}}\,da + \frac{cx^2}{Ay\sqrt{c^4+a^4}}\left\{c^2-\sqrt{(a^4+c^4)}\right\}dc \tag{9b}$$

We now have to derive the values of the scale factors h_1, h_2 and h_3.

Now $$h_1^2 = \left\{\frac{a^3(c^2-x^2)}{Ax\sqrt{(c^4+a^4)}}\right\}^2 + \left\{\frac{a^3(c^2+y^2)}{Ay\sqrt{(c^4+a^4)}}\right\}^2$$

or finally $$h_1 = \frac{a^3}{A^{1/2}\left((c^4+a^4) - c^2\sqrt{(c^4+a^4)}\right)^{1/2}} \tag{10a}$$

Similarly $$h_2 = \frac{1}{A^{1/2}\left(c^4+a^4\right)^{1/4}}\left\{\sqrt{(a^4+c^4)}-c^2\right\} \tag{10b}$$

and $$h_3 = 1 \tag{10c}$$

Now the general form of the Laplacian operator, in terms of the scale factors h_1, h_2 and h_3, is given as

$$\nabla^2 = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial a}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial a}\right) + \frac{\partial}{\partial c}\left(\frac{h_1h_3}{h_2}\frac{\partial}{\partial c}\right) + \frac{\partial}{\partial z}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial z}\right)\right\} \tag{11}$$

It can be found that

$$\frac{1}{h_1h_2h_3} = \frac{A\left(c^4+a^4\right)^{1/2}}{a^3\left(\sqrt{(c^4+a^4)}-c^2\right)^{1/2}} \tag{12}$$

and finally the Laplacian operator will have the form

$$\nabla^2 = \frac{3Ac^2}{a^7}\left\{\sqrt{(a^4+c^4)}-c^2\right\}\frac{\partial}{\partial a} + \frac{A}{a^6}\sqrt{(a^4+c^4)}\left\{\sqrt{(a^4+c^4)}-c^2\right\}\frac{\partial^2}{\partial a^2}$$
$$+ \frac{3Ac}{\left(\sqrt{(c^4+a^4)}-c^2\right)^2}\frac{\partial}{\partial c} + \frac{A\sqrt{(c^4+a^4)}}{\left(\sqrt{(c^4+a^4)}-c^2\right)^2}\frac{\partial^2}{\partial c^2} + \frac{\partial^2}{\partial z^2} \tag{13}$$

Replacing a by ρ and c by ξ, the final wave equation of the system will be as follows :

$$\left[\frac{3A\xi^2}{\rho^7}\left\{\sqrt{\rho^4+\xi^4)}-\xi^2\right\}\frac{\partial}{\partial\rho} + \frac{A}{\rho^6}\sqrt{\rho^4+\xi^4)}\left\{\sqrt{\rho^4+\xi^4)}-\xi^2\right\}\frac{\partial^2}{\partial\rho^2}\right.$$
$$\left. + \frac{3A\xi}{\left(\sqrt{(\xi^4+\rho^4)}-\rho^2\right)^2}\frac{\partial}{\partial c} + \frac{A\sqrt{(\xi^4+\rho^4)}}{\left(\sqrt{(\xi^4+\rho^4)}-\xi^2\right)^2}\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial z^2}\right]\psi$$
$$+ \left(n^2k^2-\beta^2\right)\psi = 0 \tag{14}$$

where ψ is the wave function, and n, β and k are the refractive indix, z-component of the propagation vector and the free-space propagation constant respectively.

Now we can consider two cases :

Case I : $\rho/\xi \ll 1$, *i.e.* the corner region of the guide.

Applying this approximation eq. (14) will be reduced to the form

$$\left[4\sqrt{2}\xi^6\frac{\partial^2}{\partial\xi^2} + \frac{\rho^6}{\sqrt{2}}\frac{\partial^2}{\partial\rho^2} + 12\sqrt{2}\xi^5\frac{\partial}{\partial\xi} + \frac{3}{\sqrt{2}}\rho^5\frac{\partial}{\partial\rho}\right]\psi$$
$$+ (n^2k^2 - \beta^2)\rho^6\psi = 0 \tag{15a}$$

Case II : $\rho/\xi \gg 1$, *i.e.* the central region of the guide.

This approximation reduces eq. (14) to the form

$$\left[\frac{\partial^2}{\partial\rho^2} + 3\frac{\partial^2}{\partial\xi^2} + 3\xi\frac{\partial}{\partial\xi}\right]\psi + (n^2k^2 - \beta^2)\psi = 0 \tag{15b}$$

We shall consider the *case I* only as we are interested to solve the wave equation in the corner regions. Therefore, after separating the variables eq. (15a) can be broken into two parts as

$$4\sqrt{2}\xi^6\frac{1}{\psi(\xi)}\frac{\partial^2}{\partial\xi^2}\psi(\xi) + 12\sqrt{2}\xi^5\frac{1}{\psi(\xi)}\frac{\partial}{\partial\xi}\psi(\xi) = \alpha \tag{16a}$$

$$\frac{1}{\sqrt{2}}\rho^6\frac{1}{\psi(\rho)}\frac{\partial^2}{\partial\rho^2}\psi(\rho) + \frac{3}{\sqrt{2}}\rho^5\frac{1}{\psi(\rho)}\frac{\partial}{\partial\rho}\psi(\rho) + (n^2k^2 - \beta^2)\rho^6 = -\alpha \tag{16b}$$

In these equations α is a constant.

Equation (16a) is insignificant as it does not contain the parameter β. Also, in eq. (16b) the order of $(n^2k^2 - \beta^2)$ is very smaller as compared to α/ρ^6. Thus, eq. (16b) can be reduced to the form

$$\frac{d^2\psi}{d\rho^2} + \frac{3}{\rho}\frac{d\psi}{d\rho} + \frac{\alpha\sqrt{2}}{\rho^6}\psi = 0 \tag{17}$$

which is a form of the Bessel equation. From eq. (17) the solution in the core and the cladding regions of the guide would be derived as

$$\psi_{\text{core}} = \sqrt{(2/\pi\gamma_1)}\sin(F) - \sqrt{}(2\gamma_1)\xi_1\int C(\sqrt{F})F^{-3/2}dF$$
$$+ \sqrt{}(\gamma_1/\pi)\xi_1\int\frac{\sin F}{F^2}dF \tag{18}$$

$$\psi_{\text{clad}} = \sqrt{(2/\pi\gamma_1)}\sin(F) - \sqrt{}(2\gamma_1)\xi_2\int C(\sqrt{F})F^{-3/2}dF$$
$$+ \sqrt{}(\gamma_1/\pi)\xi_2\int\frac{\sin F}{F^2}dF \tag{19}$$

where $\xi_1 = n_1^2k^2 - \beta^2$, $\xi_2 = \beta^2 - n_2^2k^2$, $\gamma_1 = \sqrt{(\alpha\sqrt{2})}$, $F = \gamma_1 / \rho^2$ and $C(\sqrt{F})$ is the cosine integral function of $\sqrt{F}$. Also n_1 and n_2 are the refractive indices of the core and the cladding regions of the guide with $n_1 > n_2$.

Equations (18) and (19) are the wave functions for the core and the cladding regions of the guide having a Piet Hein geometrical cross section.

Further work in this direction is in progress and will be reported in a later communication.

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